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LETTER TO THE EDITOR

Nature of the singularity in a spin-glass model

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Abstract. High temperature series expansions are developed for the specific heat of a random bond Ising model of a spin glass for standard two- and three-dimensional lattices. Padé approximant analysis of the series indicates the absence of any singularity on the positive real axis. The solution for the Bethe lattice is investigated using results obtained previously for the Mattis random site model. It is concluded that the high temperature partition function and all its derivatives with respect to magnetic field have no singularity at the transition temperature. This behaviour may also extend to lattice models.

1. Introduction

We consider an Ising random bond model of a spin glass (Edwards and Anderson 1975) in which the interactions have equal probability of being $\pm J$. It is assumed that only nearest-neighbour interactions are non-zero. The development of series expansions for the model was discussed in general terms by Domb (1976) who showed that in the high temperature phase all spin-pair correlations are zero, and the magnetic susceptibility corresponds to uncoupled spins. However, the specific heat and the even derivatives of the susceptibility do show cooperative effects.

Using the method described in Domb (1976) we have derived high temperature expansions for the specific heat of this model for a number of lattices in two and three dimensions. Writing the partition function for the model in zero field in the form

$$\overline{\ln Z_N(\beta, 0)} = N \ln 2 + \tfrac{1}{2}Nq \ln \cosh \beta J + \sum_n a_{2n} w^{2n} \qquad (w = \tanh \beta J), \tag{1}$$

where the bar denotes a stochastic average, we list in table 1 the values of a_{2n} for the simple quadratic (SQ), triangular (T), simple cubic (SC), body-centred cubic (BCC) and face-centred cubic (FCC) lattices. It will be seen that the behaviour of the coefficients is very irregular, and no clear pattern emerges even in regard to the oscillations in sign.

2. Specific heat

The corresponding specific heat series derived from (1) is of the form

$$C_H/Nk = (\beta J)^2 \sum_n b_{2n} w^{2n}, \tag{2}$$

† Work performed while on leave of absence at Bar-Ilan University, Ramat Gan, Israel.

Table 1. Coefficients a_{2n} in the zero-field partition function expansion.

n	T	FCC	SQ	SC	BCC
3	-1	-4	0	0	0
4	$-1\frac{1}{2}$	$-16\frac{1}{2}$	$-\frac{1}{2}$	$-1\frac{1}{2}$	-6
5	3	-12	0	0	0
6	$11\frac{1}{2}$	291	-1	-11	-50
7	-12	1956	4	36	384
8	$107\frac{1}{4}$	$-207\frac{3}{4}$	$-6\frac{3}{4}$	$-83\frac{1}{4}$	-900
9	$-71\frac{1}{3}$	$-86893\frac{1}{3}$	24	656	9384
10	$391\frac{1}{2}$	-550140	-74	-2250	-57524
11	4017	891336	212	9996	253592

where the coefficients b_{2n} are listed in table 2. These series were subjected to Padé approximant analysis using a computer program provided by Dr C J Pearce. The important feature to emerge was the *absence of any singularity on the positive real axis.* For loose packed lattices there is a non-physical singularity given by

$$w^2 \sim -1/\mu \tag{3}$$

where μ is the self-avoiding walk limit; but there is no evidence of a singularity in the neighbourhood of

$$w \sim 1/\mu^{1/2} \tag{4}$$

as given by the self-avoiding polygon term (Domb 1976). It seems that higher-order graphs whose contributions are alternately negative and positive remove this singularity.

The results of the Padé approximant analysis for different lattices are shown in figures 1 and 2. There is a clear difference in behaviour between the closely packed lattices (T, FCC and BCC) which show a characteristic maximum and become negative at sufficiently low temperatures, and the loose packed lattices for which there is a fairly steady increase as the temperature is lowered.

Table 2. Coefficients b_{2n} in the specific heat series.

n	T	FCC	SQ	SC	BCC
0	3	6	2	3	4
1	-3	-6	-2	-3	-4
2	-30	-120	0	0	0
3	-12	-636	-28	-84	-336
4	420	864	64	192	768
5	810	39624	-168	-1560	-7032
6	-5166	270864	1016	9720	84288
7	-19242	-771216	-3344	-35808	-374328
8	30564	-26072232	11640	250920	3412944
9	165822	-152802828	-45508	-1302732	-28184752
10	1518258	822191712	165352	6642504	166388032

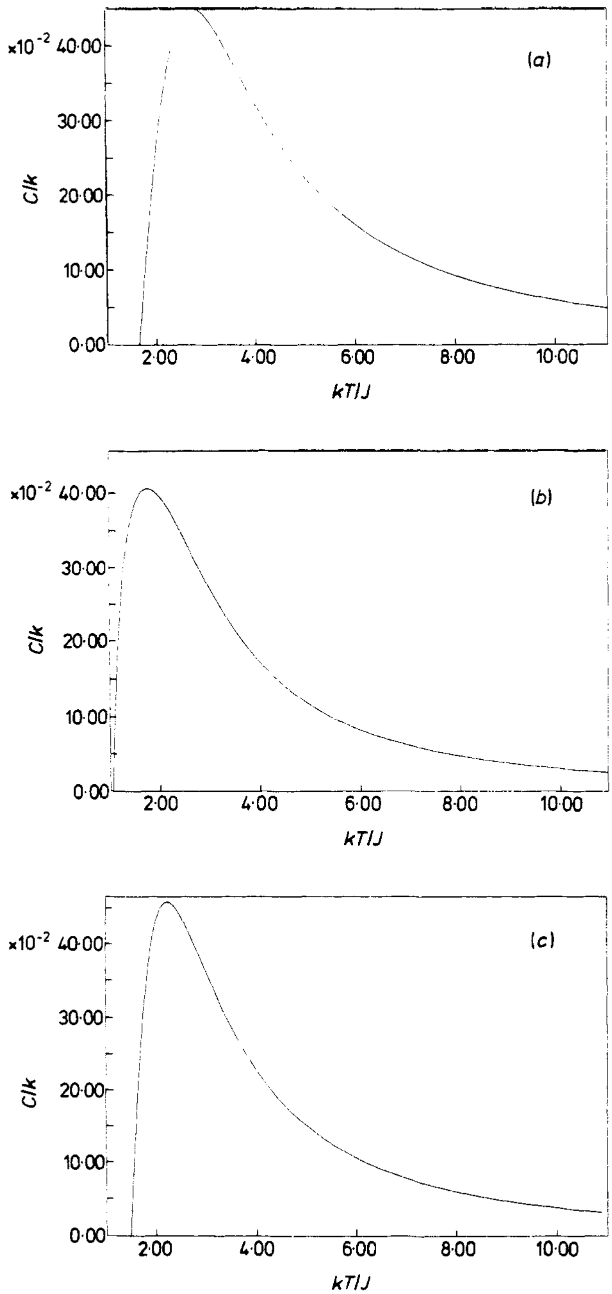

Figure 1. Padé approximant estimates of the specific heat of the high temperature phase for close-packed lattices: (*a*) FCC; (*b*) T; (*c*) BCC.

3. Derivatives of susceptibility

In regard to the second derivative of the susceptibility, Dr A P Young has pointed out that the discussion in Domb (1976) contains an error. In the calculation of $\langle \sigma_i \sigma_j \rangle^2$ it is

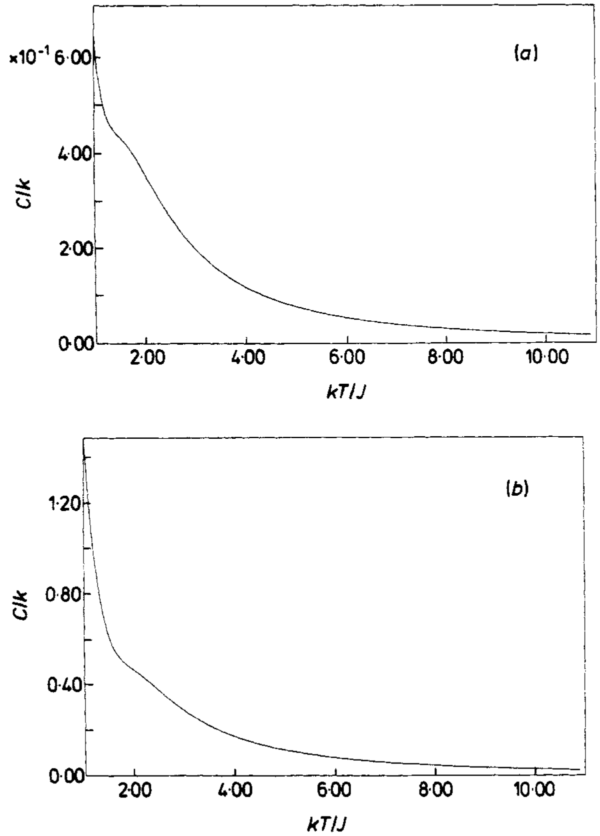

Figure 2. Padé approximant estimates of the high temperature phase for loose-packed lattices: (*a*) SQ; (*b*) SC.

possible for different pair correlation graphs which overlap to give a non-zero contribution as illustrated in figure 3; hence the conclusion that $\chi_0^{(2)}$ has a singularity at $w = w_c^{1/2}$ given in the above paper is invalid. It is possible that the effect of these neglected terms is to remove the singularity as in the case of the specific heat. The point requires further investigation.

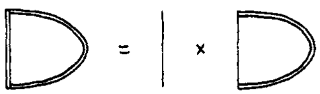

Figure 3. Overlapping of correlation graphs to give non-zero contributions.

4. Bethe lattice and Mattis model

A number of investigations have referred to the solution for a spin glass on a Bethe lattice, but to the best of our knowledge no detailed description has yet been given of the nature of the transition. (Except in the one-dimensional case (Grinstein *et al* 1976), for which there is no transition at non-zero temperature.) The Mattis model of random site spins (Mattis 1976) can be investigated exactly (Bideaux *et al* 1976) and for the Bethe lattice the random bond model is identical with this random site model. This can be seen at low temperatures by the one-to-one correspondence between excited states for the two models, and at high temperatures by the identity of the configurational structure of the high temperature series expansions.

In fact for the Bethe lattice with no closed circuits the high temperature description given in Domb (1976) is valid (as pointed out by Young and Stinchcombe 1976). Hence we have the following behaviour of the high temperature functions for the spin glass model.

(*a*) The zero-field partition function $\ln Z_N$ is identical with that of the standard Ising model $\ln Z_N^{\mathrm{I}}$, and has no singularities.

(*b*) The zero-field susceptibility corresponds to uncoupled spins and has no singularity.

(*c*) The second derivative of the susceptibility is related to the zero-field susceptibility of the standard Ising model, and has a singularity at $w_c = 1/(q-1)^{1/2}$ ($w = \tanh \beta J$, $q =$ coordination number of lattice).

(*d*) For higher derivatives if r is odd the $2r$th derivative of $\ln Z_N$ is identical with that for uncoupled spins; whilst if r is even the $2r$th derivative of $\ln Z_N$ can be related to the rth and lower derivatives of $\ln Z_N^{\mathrm{I}}$. The first singularity is given by $w_c = 1/(q-1)^{1/2}$, and there will be other singularities at lower temperatures of the form $w_c = (q-1)^{-1/n}$.

The low temperature behaviour can be derived as in Bideaux *et al* (1976), and is as follows.

(i) The zero-field partition function $\ln Z_N$ is identical with that of the standard Ising model and has no singularity at the transition point $w_c = 1/(q-1)$. The transition consists of a discontinuity in specific heat.

(ii) The zero-field susceptibility is related to the spontaneous magnetisation of the standard Ising model. It has a singularity at the transition point $w_c = 1/(q-1)$, and rises to a maximum value at this point. Note that the singularity is at $w_c = 1/(q-1)$ and not at $w_c = 1/(q-1)^{1/2}$ which arises in high temperature functions.

(iii) The second derivative of the susceptibility $\chi_0^{(2)}$ is related to the susceptibility of the standard Ising model below T_c and is therefore infinite at $H = 0$. However, if we take the limit as $H \to 0$ we will get a function with a singularity at $w_c = 1/(q-1)$. Hence the transition occurs at a higher temperature than the singularity of the high temperature second derivative, and the latter is not directly relevant to the transition.

(iv) The $2r$th derivative of $\ln Z_N$ is related to the rth derivative of $\ln Z_N^{\mathrm{I}}$. All of these derivatives therefore have singularities at $w_c = 1/(q-1)$. The singularities in the corresponding high temperature functions have no direct relevance to the transition.

We see therefore that the transition point is completely determined by the singularity in the low temperature behaviour. The high temperature functions themselves give no evidence of any transition and can be continued in a metastable state below the transition. This is an unusual pattern of behaviour not encountered in any of the other models of critical behaviour.

5. Nature of singularity in the lattice model

There is no obvious reason why the high temperature behaviour of a lattice model should be significantly different from that for a Bethe lattice. We have seen that the free energy and susceptibility in zero field have the same singularity structure, and one may reasonably expect this to extend to higher derivatives. The main difference should lie in the low temperature behaviour, since the lowest energy state for a lattice model has a finite entropy, and this may destroy the ordering (Toulouse 1977). The behaviour of the order parameter may therefore be significantly different, as was indicated in the results of Young and Stinchcombe (1976) for the SQ lattice.

But it seems unlikely that the finite entropy will result in the susceptibility reducing to that of uncoupled spins at low temperatures. This is not the case with the antiferromagnetic triangular Ising lattice which also has a finite entropy at $T=0$ (see e.g. Domb 1974). Hence we would expect a transition in susceptibility (but possibly a weak one) corresponding to the meet of the low and high temperature functions.

Series expansions for the 'order parameter susceptibility' at high temperatures were calculated by Fisch and Harris (1977) for hypercubical lattices in d dimensions. We have learned recently from a preprint that the calculations have been extended to the free energy in zero field. In the region where our results overlap with their's it is pleasing to record numerical agreement (except for a trivial factor of 2). The above authors draw the conclusion that there are no singularities for $d<4$ but that singularities and associated critical exponents appear when $d<4<6$. We wish to point out, following the behaviour of the Bethe lattice, that the existence of a singularity in the high temperature expansion does not ensure that the singularity corresponds to critical behaviour. It is necessary to establish from the low temperature behaviour that the high temperature singularity is not spurious, and corresponds to a true critical point.

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